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Principal Investigator: Gangbo, Wilfrid .
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Organization: GA Tech Res Corp - GIT
Submitted By:

Title:
FRG: Collaborative Research: Applications of Transportation Theory to Nonlinear Dynamics

Project Participants

Senior Personnel

Name: Gangbo, Wilfrid
Worked for more than 160 Hours: Yes
Contribution to Project:

Name: Evans, Lawrence
Worked for more than 160 Hours: Yes
Contribution to Project:

Name: Caffarelli, Luis
Worked for more than 160 Hours: Yes
Contribution to Project:

Name: McCann, Robert
Worked for more than 160 Hours: Yes
Contribution to Project:

Name: Feldman, Mikhail
Worked for more than 160 Hours: Yes
Contribution to Project:

Name: Cullen, Mike
Worked for more than 160 Hours: Yes
Contribution to Project:

Post-doc

Graduate Student

Undergraduate Student

Technician, Programmer

Other Participant

Research Experience for Undergraduates

Organizational Partners

Other Collaborators or Contacts

Wilfrid Gangbo has been collaborating with L. Ambrosio, G. Bouchitte, J.P. Seppecher, B. Dacorogna, O. Savin, M. Chipot, B. Kawohl, G. Pisante, M. Cullen, T. Pacini, A. Tudorascu, T. Nguyen, V. Oliker, M. Westdickenberg, HK. Kim.

Robert McCann has collaborated with Yong-Jung Kim, Dejan Slepcev, Luis Caffarelli, Jochen Denzler, Peter Topping, Adam Oberman, Jose A. Carrillo, Cedric Villani, Dario Cordero-Erausquin, Michael Schmuckenschlaeger, Dorian Goldman, Maxim Trokhimtchouk, Pierre-Andre Chiappori, Lars Nesheim, Hwakil Kim, Daniel Mattes, Giuseppe Savare, Najma Ahmad.

LC. Evans has collaborated with W. Gangbo, O. Savin, J. Dorfman, J. Neu, M. Zworski.

Mikhael Feldman has collaborated with Gui-Qiang Chen,

L. Caffarelli has collaborated with Robert McCann, P. Souganidis,

Senior scientist, Mike Cullen, listed on the proposal, has been collaborating with Douglas, M. Sewell and I. Roulstone, M Wakefield.

Activities and Findings

Research and Education Activities:

During 2004-05, W. Gangbo has been training his postdoc G. Pisante, his graduate students, T. Yolcu and HK Kim to learn the Monge-Kantorovich theory and the theory of Michell trusses. He has been teaching a special topic class during Spring 2005, on the link between the Monge-Kantorovich theory and Stochastic Differential Equations. He has given four lectures in the Carnegie Mellon 2004 Summer School, on the Monge-Kantorovich theory. In 2005-06, W. Gangbo has interacted with his postdocs A. Tudorascu, T. Nguyen, G. Pisante, his graduate students, T. Yolcu and HK Kim on problems related to the Monge-Kantorovich theory. In a work in progress, PhD student HK Kim has proven convergence to the infinite dimensional Hamiltonian ordinary differential equations, starting from a finite dimensional one. He is investigating uniquness properties of Hamiltonian flows.

During 2004-2006, M.Feldman has been training his graduate students M. Bae and N. Panike and postdoc J. Chen on Monge-Kantorovich theory and nonlinear partial differential equations. Funds of FRG were also used by M.Feldman to support visits to Madison of G.Loeper and F.Ferrari.

Active collaborations between Robert McCann and his graduate students and postdocs with US based scientists and members of the FRG have been maintained with the support of the grant. This includes McCann?s extended visits to the University of California at Los Angeles (August 2004) and the University of Texas at Austin (March - May 2005). These collaborations focus on (i) long time asymptotics for nonlinear diffusion; (ii) exact solutions to the semigeostrophic theory, and (iii) optimal transportation with unequal masses or unusual costs. Active collaborations between Robert McCann and his graduate students and postdocs with US based scientists and members of the FRG have been maintained with the support of the grant. This includes McCann?s extended visits to the University of Texas at Austin (March - May 2005) and the Mathematical Sciences Research Institute at the University of California at Berkeley (August - December 2005). These collaborations focus on (i) long time asymptotics for nonlinear diffusion; (ii) exact solutions to the semigeostrophic theory, and (iii) optimal transportation with unequal masses or unusual costs.

Funds of the FRG have partially been used by L. Caffarelli to support two PhD students: Luis Silvestre and Emmanuel Milakis.

Evans's very good PhD student Y. Yu has now finishing his thesis, mostly on the 'Aronsson equation'. Mr. Yu has been supported this semester on funds remaining from our previous FRG grant. He is currently holding a postdoc position at the University of Texas, to work with Caffarelli and Souganidis.

A book, authored by senior scientist M. Cullen, 'A mathematical theory of large-scale atmospheric flow' has been completed by a lecture note series to be published by World Scientific Press.

Active collaborations between Robert McCann and his graduate students and postdocs with US based scientists and members of the FRG have been maintained with the support of the grant. This includes McCann?s extended visits to Austin (March - May 2005) and Berkeley (August - December 2005), and upcoming visits to the Mathematical Sciences Research Institute at the University of California at Berkeley (April - May and August 2007), the FRG conference in
Edinburgh July 2007, and to the Institute for Pure and Applied Mathematics at the University of California at Los Angeles (April - June 2008). These collaborations focus on
(i) regularity of optimal mappings with unequal masses or unusual costs,
(ii) applications of optimal transportation to economics
(iii) second and fourth order parabolic and geometric flows;
(iv) semigeostrophic dynamics of the atmosphere

{\bf A. The infinity Laplacian.}

O. Savin and LC EVans have finally finished their paper showing $C^{1, \alpha}$ regularity for weak solutions of the infinity Laplacian PDE
$$
\Delta u := \sum_{i, j =1}^{n} u_{x_i}u_{x_j}u_{x_ix_j}= 0
$$
in two dimensions.
In spite of many attempts they still cannot extend their proof to higher dimensions, and have for the time being given up trying.

In collaboration with Y. Li, L. Caffarelli made progress on
multiple valued solutions of the Monge Ampere equation. This research is related to singularities of metrics related to a convex potential, with some particular affine invariance, and was motivated by questions of Tian and Yau

During 2005-08, W. Gangbo has completed a research project with his postdocs A. Tudorascu and T. Nguyen. He is working with his graduate students, T. Yolcu, HK Kim. He is supervising Sedjro who is in the PhD program at GTech, for a reading course. This supervision has lasted four semesters. Gangbo has lectured on a special topic class on gradient flows in metric spaces. The class was attended by 7 faculty members and 7 students. Gangbo give a ten hours lecture in Benin during summer 2006 and the same lecture in Dakar, Senegal that summer. He organized a conference in Benin in September 2007.

Cullen made a 2 week visit to Wisconsin under the FRG programme. At this visit, useful progress was made in formulating the semi-geostrophic problem with variable Coriolis parameter described by Cullen,M.J.P., Douglas,R.J., Roulstone,I. and Sewell, M.J. (J. Fluid Mech, 531, 123-157.). It appears essential to control the time rate of change of the Lagrangian map following almost all particles. In the shallow water problem with constant rotation, a formal argument for achieving this can be constructed, which would also prove uniqueness in that case. This argument does not work in the incompressible case. A formal procedure for solving the problem with variable rotation rate has been found, but there are still significant issues in justifying it rigorously.

During 2006-2008 Feldman was working, with his postdoc J. Chen and gradient student N. Panike, on semigeostrophic (SG) system, focusing on the following questions: existence of Lagrangian solution for shallow water SG model with variable Coriolis parameter and on sphere, and regularity properties of weak solutions. Some progress has made during a visit of Cullen to Madison. Also, Feldman worked with F. Ferrari (Bologna) on existence of optimal map for Monge problem
with cost=distance on Heisenberg group.

Findings:

W. Gangbo, in a joint work with G. Bouchitte and M. Seppecher have introduced new tools to produce for the first time, existence of Michell trusses for a general class of forces. These trusses are still obtained in a weak sense and the results obtained leave many open questions. In a joint work with C. Evans and O. Savin, W. Gangbo studied a very instructive example which illustrates how some hard

In the semigeostrophic fluid model for atmospheric and ocean dynamics, McCann and Adam Oberman (a postdoctoral fellow at UT Austin, now at Simon Fraser University) discovered a small family of exact Hamiltonian dynamics corresponding to quadratic pressure and stream function solutions the two dimensional incompressible theory. These solutions display some surpising features which enhance our understanding of the meteorology and oceanography of this regime. McCann is continuing to study the stability of these solutions with Daniel Spirn (U Minnesota).

Finally, McCann continues to investigate basic existence, uniqueness, and regularity properties of optimal mappings. With Najma Ahmad (PhD ?04), he considered developing a general uniqueness theory for costs which change concavity.

For fast nonlinear diffusion, Denzler (U Tenn Knoxville) and McCann began a new collaboration with Herbert Koch (UC Berkeley / MSRI) to show the full asymptotics governing convergence to self-similarity in the nonlinear problem are correctly predicted by the linearization carried out earlier by Denzler and McCann. The leading order terms in this expansion were verified by McCann in separate works with postdoctoral fellows Yong Jung Kim (now at the Korean Advanced Institute for Science and Technology) and Dejan Slepcev (now at Carnegie Mellon University). In all of these works, Vazquez uniform convergence in relative error plays a key role in establishing a rigorous connection between the nonlinear dynamics and its linearization.

Together with Luis Caffarelli (UT Austin), McCann has developed a theory for incomplete transportation, in which only a fraction of the available production capacity and demand are utilized, this fraction being selected to minimize transportation cost. In particular, they have succeeded in establishing differentiability of the free boundary separating the active from inactive mines, and continuity of the optimal map up to this boundary, assuming quadratic transportation costs.

Finally, with Maxim Trokhimtchouk (UC Berkeley PhD candidate), McCann has developed an existence and uniqueness theory for the economic problem of dividing a continuous labor force into manager and assistant pairs, where each employee must be assigned a role so as to optimize productivity. Subdividing the labor force into manager and assistant roles amounts to another kind of free boundary problem in optimal transportation.

Under the supervision of L. Caffarelli, Silvestre and Milakis completed a project on Optimal regularity for the Neuman problem for fully non linear equations. This largely superseeds previous work by Lions and Trudinger. They get optimal theorems with minimal hypotesis. Milakis also studied the problem of minimizing transportation cost into an unknown domain plus the perimeter of its boundary. He showed Holder differentiability of the boundary of the unknown domain, except for singularities of lower dimension. Caffarelli is completing a project in collaboration with P Souganidis, on rate of convergence of solutions to fully non linear equations in random media to their ergodic limit, in case of a given mixing rate. They are planning to make a connection between their work the Monge Ampere equations associated with optimal transportation.

Inspired by ideas of L.~Prigozhin, Evans has introduced the singular limit problem

$$ \begin{cases}
u_t^{\epsilon} - \mbox{div} \left(\frac {\phi(u^{\epsilon}-g)}{\epsilon} D_xu^{\epsilon}\right) = f \quad \mbox{ in ${\mathbb R}^2 \times (0,\infty)$.} \\
\qquad \qquad \qquad \qquad \ \ \ u^{\epsilon} = g \quad \mbox{ on ${\mathbb R}^2 \times \{ t = 0 \}$.}
\end{cases}
$$

Here g represents the height of a given landscape and $\phi(x) = 0$ if $x \le 0$, $\phi(x) > 0$ if $x > 0$. If $u^{\epsilon} \to u$ as $\epsilon \to 0$, then formally

$$
|Du| = 0 \quad \mbox{ on the set } \{u > g\}.
$$

Think of f as a ``rainfall" rate and $u$ the height of the resulting ``water flow" over the landscape. The components of the set $\{u > g\}$ are ``lakes".

This is a form of the ``sandpile" models introduced by Aronsson, Evans and Wu some years ago in connection with optimal mass transport. Evans has recently discovered how to use a variant of the Baiocchi transform from free boundary theory to study the sets $\{u^{\epsilon} > g \}$, and hopes thereby to make rigorous some formal asymptotics J. Neu worked out.

Evans and his student J. Dorfman have been working on theoretical and numerical studies of the singular limit problem
$$ \begin{cases}
u_t^{\epsilon} - \mbox{div} \left(\frac {\phi(u^{\epsilon}-g)}{\epsilon} D_xu^{\epsilon}\right) = f \quad \mbox{ in ${\mathbb R}^2 \times (0,\infty)$.} \\
\qquad \qquad \qquad \qquad \ \ \ u^{\epsilon} = g \quad \mbox{ on ${\mathbb R}^2\times \{ t = 0\}$.}
\end{cases}
$$
Here g represents the height of a given landscape and
$\phi(x) = 0$ if $x \le 0$, $\phi(x) > 0$ if $x > 0$.
If
$u^{\epsilon} \to u$ as $\epsilon \to 0$, then
formally
$$
|Du| = 0 \quad \mbox{ on the set } \{u > g\}.
$$
One thinks of f as a ``rainfall" rate and u the
height of the resulting ``water flow" over the landscape. The components of the set $\{u > g\}$ are ``lakes". They have also rigorously proved that one form of the BCRE models for sandpiles converges in an appropriate asymptotic limit to the earlier sandpile models of Aronsson-Evans-Wu and Prigozhin. This work will be contained in Dorfman's PhD thesis, and probably a future joint paper.

Evans's PhD student, Y. Yu has finishing his thesis, mostly on
the {\em Aronsson equation}
$$
A_H[u] := - \sum_{i, j =1}^{n} H_{p_i}H_{p_j}u_{x_ix_j}- \sum_{i =1}^{n}
H_{p_i}H_{x_j} = 0,
$$
where the nonlinearity H is evaluated at (Du,x), and its connections with weak KAM, especially the Mather and Aubry sets.

{\bf A. The infinity Laplacian.}
O. Savin and Evans have intensively worked last fall trying to prove $C^{1, \alpha}$ regularity for solutions of the infinity Laplacian PDE
$$
\Delta u := \sum_{i, j =1}^{n} u_{x_i}u_{x_j}u_{x_ix_j}= 0.
$$
They have devised a plausible scheme for deriving the necessary estimates from a highly unusual blow-up procedure with a highly nonisotropic rescaling, leading to the new PDE
$$
\sum_{i, j =1}^{n-1} v_{x_i}v_{x_j}v_{x_ix_j} + 2 \sum_{i=1}^{n} v_{x_i}v_{x_ix_n} + v_{x_n x_n} = 0.
$$
The method depends upon their finding interior gradient bound for bounded solutions v, which we have done only for $n =2$ dimensions. They are intensely studying this problem, but have in spite of many attempts have failed to derive the requisite estimates.
They are writing a joint paper documenting their progress so far.

Evans's student S. Armstrong, who was supported by FRG funds last fall, is working with him on the periodic homogenization problem of studying the limit as $\epsilon \to 0$ of the PDE
$$ A_{H^ \epsilon}[u^\epsilon] := \sum_{i, j =1}^{n} H_{p_i}H_{p_j}u^\epsilon_{x_ix_j}-
\frac 1 \epsilon \sum_{i =1}^{n} H_{p_i}H_{x_j} = 0, $$
where the nonlinearity H is evaluated at $(Du^\epsilon, x / \epsilon)$.They have a formal proof that the limit PDE is
$$ A_{\bar H}[u] := \sum_{i, j =1}^{n}\bar H_{p_i}\bar H_{p_j}u_{x_ix_j} = 0,$$
$\bar H$ denoting the effective Hamiltonian from weak KAM theory, and are trying to devise a rigorous proof.

With R. Douglas, M. Sewell and I. Roulstone, M. Cullen studied Generalised semi-geostrophic theory on the sphere. The issues requiring further rigorous analysis were clarified in preparing the final version. Though there are a number of these, the formal arguments used in the paper are solid, and they are confident that a rigorous treatment will be possible. A paper 'Lagrangian solutions of semi-geostrophic equations in physical space' by M. Cullen and M. Feldman is under review by SIAM J Appl Math. Computations using a semi-geostrophic model to describe the effect of spatially varying surface forcing on the tropical circulation in the atmosphere have been carried out. The basic mechanism

is successfully simulated. The next stage of the work is to compare the time-averaged circulation with that predicted by conventional climate models. This area is a weakness of current climate models, and is very important for simulating the atmospheric general circulation. A paper describing the initial computations, 'Modelling the response of the atmosphere to equatorial forcing' by M Wakefield and M. Cullen has been accepted by the International Journal of Numerical Methods in Fluids.

In collaboration with Yanyan Li, L. Caffarelli study singularities of metrics related to a convex potential, with some particular affine invariance. This work was motivated by questions of Tian and Yau.

The FRG supported M Milakis (and advanced graduate student at Crete at the time, that came for a year and has now accepted an Instructor position at U of Washington). Under the direction of L. Caffarelli,
he studied the regularity of the free boundary in transportation when the transportation cost plus the perimeter of the image is minimized. He showed that the Almgreen Tamanini theory could be applied.
In collaboration with another student, Luis Silvestre (supported by Caffarelli's individual NSF) they gave a complete description of the boundary regularity for the Neumann problem for viscosity solutions of fully non linear equations

Yannick Sire, a postdoctoral fellow from Toulouse, has been partially supported by the FRG. He has been collaborating with L. Caffarelli on free boundary problems with integral operators, and with De La Llave on dynamical systems.

As part of our ongoing exploration of the basic properties of optimal mappings, McCann and postdoctoral fellow Young-Heon Kim (U Toronto) were excited to discover that optimal transportation induces an intrinsic geometry in which curvatures govern the regularity of optimal mappings. They are currently exploring this phenomenon and using it to extend the theory of partial transportation developed by Caffarelli (UT Austin) and McCann to a broader class of cost functions. This theory addresses the question of which producers should stop producing and which consumers should stop consuming if only a limited fraction of available production capacity and demand are to be utilized.

In a new collaboration with Peter Topping (Warwick), McCann also discovered, that the reverse time Ricci flow of the metric makes diffusion a contraction with respect to Wasserstein distance on any manifold (without curvature assumptions). This contractivity can in fact be used to characterize Ricci flow, and represents an unexpected connection of optimal transportation to geometric flows. For fast nonlinear diffusion, Denzler (U Tenn Knoxville), McCann and Herbert Koch (UC Berkeley and Bonn) continued their work on to show the full asymptotics governing convergence to self-similarity in the nonlinear problem are correctly predicted by the linearization carried out earlier by Denzler and McCann. With postdoctoral fellow Ben Stephens (U Toronto), McCann carried out the analogous linearization for a family of fourth order equations including both the quantum-drift diffusion equation and the thin film equation. Denzler and McCann have also discovered a family of explicit solutions to these second- and fourth-order equations, which possess affine rather than radial symmetry, allowing diffusion rates in different geometries to be explore. With Aaron Smith and Almut Burchard (U Toronto), McCann has begun exploring the implications for asymptotics of the Yamabe scalar curvature flow equation around the cigar soliton.

McCann has also been engaged in an ongoing exploration of the semigeostrophic dynamics of the atmosphere and ocean. One direction has involved analyzing the stability of explicit elliptical vortex patch solutions. With Dorian Goldman (U Toronto), he has used a Melnikov approach to show the exact dynamics become chaotic under gentle periodic forcing with almost any period. The dynamics take place within the family of ellipsoidal patches, but contain
a subsystem conjugate to the shift map. With Dan Spirn (University of Minnesota), McCann has begun to analyze the stability of the unperturbed dynamics with respect to perturbations of the elliptical geometry. With Lopes-Filho and Nussenzveig-Lopes, McCann has initiated the exploration of vortex sheet solutions to these dynamics, where the vorticity is concentrated on a lower dimensional subspace.

With Pierre-Andre Chiappori (Columbia) and Lars Neshiem (University College Londom), McCann has recast the economic theory of hedonic pricing into a mathematical framework of optimal transportation, reducing the basic problem to a linear program. As part of this project, they have discovered new uniqueness results concerning such linear programs, partly in consultation with Najma Ahmad (U Toronto) and Hwakil Kim (Georgia Institute of Technology). With Maxim Trokhimtchouk (UC Berkeley PhD candidate), McCann has developed an existence and uniqueness theory for the economic problem of dividing a continuous labor force into manager and assistant pairs, where each employee must

be assigned a role so as to optimize productivity. Subdividing the labor force into manager and assistant roles amounts to another kind of free boundary problem in optimal transportation.

LC Evans, John Neu and Evans's student J. Dorfman have almost
completed the first of a projected two papers on theoretical and numerical studies of the singular limit problem
$$ \begin{cases}
u_t^{\epsilon} - \mbox{div} \left(\frac {\phi(u^{\epsilon}-g)}{\epsilon}
D_xu^{\epsilon}\right) = f \quad \mbox{ in ${\mathbb R}^2 \times (0,\infty)$.} \\
\qquad \qquad \qquad \qquad \ \ \ u^{\epsilon} = g \quad \mbox{ on ${\mathbb R}^2
\times \{ t = 0
\}$.}
\end{cases}
$$
Here g represents the height of a given landscape and $\phi(x) = 0$ if $x \le 0$, $\phi(x) > 0$ if $x > 0$. If $u^{\epsilon} \to u$ as $\epsilon \to 0$, then formally
$
|Du| = 0 \quad \mbox{ on the set } \{u > g\}.
$
They think of f as a ``rainfall" rate and $u$ the height of the resulting ``water flow" over the landscape. The components of the set $\{u > g\}$ are ``lakes".

The most interesting feature in the $\epsilon \to 0$ limit is the appearance of certain measures with support outside of $\{u > g \}$: these record the water flow along the rivers.

Evans has returned to a variational principle for weak KAM theory that he introduced several years ago. This corresponds to minimizing for large k the energy
\[
I_k[v] := \int_{\Bbb T^n} e^{kH(P +Dv,x) } dx
\]
among functions v with zero mean over the flat torus $\Bbb T^n$. His old paper shows that, surprisingly, the structure of weak KAM theory appears in the limit $k \to \infty$.

He recently observed that by switching to a more Lagrangian viewpoint, he can establish some more explicit assertions of ``approximate integrability" in the $k \to \infty$ limit. He is working hard now
to push this insight further.

In a joint work with his postdocs A. Tudorascu and T. Nguyen, Gangbo obtained existence of a new class of solutions in the 1d Euler-Poisson system. These solutions minimize an action and are unique when they are prescribed at time 0 and time $T< \pi.$ They conserve the Hamiltonian unlike the entropy solutions whose existence were established by other authors in prior works. With HK Kim, Gangbo is investigating pertubation of evolutive conservative system. The idea is to approximate an original nonsmooth Lagrangian by a class of smooth ones and obtain existence of solution. These solutions will lye on an orbit. Passing to the limit, one would obtain a solution for the original Lagrangian. Approximation of Lagrangians with good properties in the infinite dimensional set of probability measures is poorly understood. With his student T. Yolcu, Gangbo is studying smoothness of L convex sets, where L is a Lagrangian on a manifold. On needs to prove that the boundary of such set is of zero measure. This will make it possible to study a larger class of parabolic PDEs without imposing that the manifold is without boundary.

M. Cullem has carried out a set of computational demonstrations that solutions of the Navier-Stokes equations in situations relevant to meteorology converge to asymptotic limit solutions at the expected rate. Several of these use the semi-geostrophic model, to which existence of weak solutions has been proved by Monge-Kantorovich theory. In most cases the results confirm the a priori error estimates. This means that it should be possible to prove an a posteriori estimate of the difference between semi-geostrophic and Navier-Stokes solutions. This would extend the result obtained by several authors using the quasi-geostrophic model as the limit solution. The semi-geostrophic model is valid on large scales while the quasi-geostrophic model is not.

The exception is a simulation involving frontogenesis. In this situation it is likely that only the Lagrangian form of the semi-geostrophic equations makes sense. The expected convergence was not achieved after the time that discontinuities has formed. THis is potentially a serious issue for practical modelling. A by-product of the experiments was a demonstration that the semi-geostrophic solutions were very insensitive to

the discretisation. This strongly suggests that the solutions are unique, which is still an open theoretical question.

Feldman, Cullen, and Feldman's postdoc J. Chen and student N. Panike, worked on semigeostrophic (SG) system with variable Coriolis parameter. They found a Lagrangian form of this system, which also allows a natural definition of a dual space and appropriate equations in the dual space. For the case of variable Coriolis parameter, a map from physical to dual space is determined by a Monge-Ampere equation 'with variable coefficients', which apparently is not variational, i.e. does not correspond to a Monge-Kantorovich problem. This equation represents a condition that a map of a given form, which is a linear expression with variable coefficients involving gradient of the unknown function, pushes forward one to another given measures. If Coriolis parameter is constant, this equation reduces to the equation for Monge-Kantorovich problem with quadratic cost. They continue working in this framework, to obtain existence of Lagrangian solution of SG system. With F. Ferrari, Feldman works on existence of optimal map for Monge problem with cost=distance on Heisenberg group. One of main difficulties is that an appropriate coarea formula is not available on Heisenberg group.

In 2008 Mike Cullen visited M. Feldman and W. Gangbo for two weeks to continue working on their joint project. M. Cullen is also interacting the the graduate student M. Sedjro on problems related to tropical weather. M. Feldman and W. Gangbo has continued their collobaration at PIMS during Spring 2008. In 2009 Mike Cullen visited M. Feldman for two weeks to continue working on their joint project.

Training and Development:

Due to their interaction with W. Gangbo, G. Pisante (postdoc) and T. Yolcu, HK. Kim (PhD students), who are supported by the FRG grant, are developping their expertise in the Monge-Kantorovich theory as well as the theory of Michell Trusses. The project supported some of the trips of W. Gangbo to interact with people such as the probabilist Elton Hsu of Northwestern University. A. Tudorascu and T. Nguyen who are matured scientists in the field of the Monge-Kantorovich theory are expending their expertize to Hamiltonian ODEs. PhD student Zhao Kun learnt the Monge-Kantorovich theory. T. Pacini has been a postdoc with W. Gangbo. He has been offered a second postdoc position at the Imperial College in England for 2006-2008. He has recently accepted a permanent offer at the Scuola N. Pisa, starting in Fall 2009.

Working with M.Feldman, grad.students M.Bae and N. Panike and
postdoc J.Chen, develop expertise in Monge-Kantorovich theory and nonlinear partial differential equations. The project supported a trip of M.Feldman to SIAM Conference on Analysis of PDE, in which he gave a talk 'Lagrangian solutions of semi-geostrophic equations in physical space' and interacted with other mathematicians.

Many of Robert McCann's supported research projects have included
collaborations with graduate students, undergraduate students, and
postdoctoral fellows. His projects on nonlinear diffusion
with Jochen Denzler (U Tennessee), Dejan Slepcev (UCLA at the time, currently at CMU), and Yong Jung Kim all began when the collaborators were postdoctoral fellows at the University of Toronto. The linearization method of Denzler and McCann was also adopted by Dejan Slepcev to show stability of self-similar blow up in a 4th order thin film equation.

McCann's work on the semigeostrophic theory with Adam Oberman (Simon Fraser University) and currently Daniel Spirn (University of Minnesota) began when his collaborators were post-doctoral fellows at the University of Texas and Brown University, respectively. The McCann-Oberman paper also includes an appendix authored under McCann?s supervision by a University of Toronto undergraduate: Maxim Trokhimtchouk, now a UC Berkeley PhD student. He has just graduated and accepted a position in NYork. McCann continued to work with his former PhD student Najma Ahmad to develop a uniqueness theory for optimal transportation with costs which change concavity. Many of Robert McCann's supported research projects have included collaborations with graduate students, undergraduate students, and postdoctoral fellows. His projects on nonlinear diffusion with Jochen Denzler (U Tennessee), Dejan Slepcev (UCLA at the time), and Yong Jung Kim (KAIST) all began when the collaborators were postdoctoral fellows at the University of Toronto. His work with Maxim Trokhimtchouk (UC Berkeley) commenced when the latter was an undergraduate student research assistant at University of Toronto. McCann was also actively engaged in working with students and postdocs at the MSRI semester semester on PDE, initiating projects with Truyen Nguyen (Georgia Tech at the time, currently in Akron) on conformally invariant transportation with logarithmic costs, and with HK Kim (Georgia Tech, recently accepted an offer at the Courant Institute) on extremal doubly stochastic measures and uniqueness of transportation for general costs.

McCann taught teaching a graduate topics course on 'Optimal
Transportation and Nonlinear Dynamics' at the University of Toronto (Spring 2006).

Luis Silvestre and Emmanuel Milakis were two PhD students of L. Caffarelli whose trainings were partially supported by this grant. Yannick Sire, a postdoctoral fellow from Toulouse was partially supported by the FRG while working with L. Caffarelli.

Y. Yu, was a PhD student of L.C. Evans, who completed his PhD, has been partially supported by our previous FGR grant. S. Armstrong a PhD student of L.C. Evans, was supported by FRG funds last Fall. L.C. Dorfman is currently working on his dissertation with L.C. Evans. He has several other PhD student, Ryan Hynd, Hung Tran and Guoling Wu. He has run in the late summer, early fall of 2005, spring of 2006 and Spring 2009, a weekly student PDE seminar where these students take turns lecturing on various interesting topics.

Many of Robert McCann?s supported research projects have included
collaborations with graduate students, undergraduate students, and
postdoctoral fellows. His recent work on regularity of optimal mappings and free boundary problems is joint with Young-Heon Kim, a postdoctoral fellow at the University of Toronto (at the time), and he mentored Alex Bloemendale, a Master's student, in that area. His projects on nonlinear diffusion began when Jochen Denzler (U Tennessee) was a postdoctoral at the University of Toronto, and now continues with Aaron Smith (Queen's University undergraduate) and Ben Stephens (University of Toronto postdoctoral fellow) His work with Maxim Trokhimtchouk (UC Berkeley) commenced when the latter was an undergraduate student research assistant at University of Toronto. And his work on stability in the semigeostrophic system has been largely carried out with Dorian Goldman, a University of Toronto undergraduate soon to be commencing graduate studies.

Evans's student Scott Armstrong has worked with him on the periodic homogenization problem of studying the limit as $\epsilon \to 0$ of the PDE
$$
A_{H^ \epsilon}[u^\epsilon] := \sum_{i, j =1}^{n} H_{p_i}H_{p_j}u^\epsilon_{x_ix_j}-
\frac 1 \epsilon \sum_{i =1}^{n} H_{p_i}H_{x_j} = 0,
$$
where the nonlinearity H is evaluated at $(Du^\epsilon, x / \epsilon)$. The formal homogenization proof we have had for a long time now suggests a rigorous study of the linearization
\[
Lw = \frac d {dt} A_H[u+tw] |_{t=0},
\]
and they have worked out various identities and energy estimates using the operator L.

\medskip
Evans's student Maxim Trokhimtchouk has investigated the interesting system
\[
{\bold u}_t = \Delta \nabla \phi ({\bold u}),
\]
where ϕ is convex. This is a very natural generalization of scalar porous media type equations. Evans's more recent student Ryan Hynd read about partial regularity theory for the Navier Stokes equations, and Guoling Wu read about stochastic optimization, games and PDE.

The FRG supported M Milakis (and advanced graduate student at Crete at the time), that came to Austin, TX for a year, under the supervision of L. Caffarelli. He has then accepted an Instructor position at U of Washington. Among his work, he studied the regularity of the free boundary in transportation when the transportation cost plus the perimeter of the image is minimized. He showed that the Almgreen Tamanini theory could be applied. In collaboration with another student, Luis Silvestre (supported by Caffarelli individual NSF) they gave a complete description of the boundary regularity for the Neumann problem for viscosity solutions of fully non linear equations. The FRG has been partially supporting also Yannick Sire, a postdoctoral fellow from Toulouse (also with partial funding from France, and the Math Department at Austin). He has been collaborating with Caffarelli on free boundary problems with integral operators, and with De La Llave on dynamical systems.

T. Nguyen is a postdoc mentored by Gangbo. They have completed a long paper on Euler-Poisson system with another postdoc, A. Tudorascu. Truyen has been offered several position including a postdoc in Canada. He is currently on a tenure-track position in Canada. Tudorascu continued his collaboration with Gangbo. He is currently a visiting faculty member at the University of Wisconsin at Madison.

J. Chen was a postdoc mentored by Feldman. He and graduate student
N. Panike worked with Feldman on semigeostrophic system.

The FRG will held a workshop in Edinburgh in July and Cullen is one of the main organizer being in the UK. A large number of good participants have indicated acceptance. The organization is well advanced.

Truyen Nguyen was a postdoc supported by the FRG and the school of Math at Georgia Tech. Truyen received two-year postdoc offers from McGill and McMaster in Canada after working with W. Gangbo at Georgia Tech. He also received a tenure-track offer from Akron, Ohio. He is

currently holding a tenure track position in Akron, Ohio.

The FRG RAs by the FRG were match by the school of math of GTech. This made its impact higher.

Outreach Activities:

In Feb 2005, W. Gangbo gave a lecture at the Spencer High School, a minority high school in Columbus Georgia. A brief description of the Monge-Kantorovich problem and its link with real life was provided to a body of 150 students. The positive impact math could have on their every day life was illustrated with exampples such as the best strategies for saving funds for college education. W. Gangbo has been talking to Dr. Kuzman Adzievski, who is a faculty member at SCSU, a HBCU in South Carolina. Dr. Adzievski attend the workshop organized by L.C. Evans, C. Guitierrez, W. Gangbo in Berkeley in Fall 2005 and decided after that to learn more on the Monge-Kantorovich theory. He has spent an afternoon at GTech to discuss possible problems to work on. Gangbo co-organized a semester of emphasis at IPAM in Spring 2008. L.C. Evans co-organized a workshop during the first week of that program.

The Monge-Kantorovich theory, the main theme of this proposal, was a significant part of the PDE semester at MSRI Berkeley in Fall 2005. That semester ended with a workshop on the topic, organized by L.C. Evans, C. Guitierrez, W. Gangbo. The conference was held at the International House in Berkeley. The staff of MSRI provided logistical support, although the funding was provided by our FRG grant.

The highlight of this conference was a series of 5 expository talks by Robert McCann, who at the last minute replaced Cedric Villani, who had a medical emergency.

Gangbo gave a ten hour lecture at the Cheik Anta Diop in Dakar, Senegal in summer 2006. He gave the same lecture at the Institute of Mathematical Sciences and Physics in Porto-Novo, Benin, the same summer. He is giving a special topic class to 7 faculty members and 7 students at his host institution. He has mentored the research of Y. Olubummo from Spelman College during summer 2006. He has given a colloquium talk at the Middle Tennessee State University which has a 5 year program (no PhD program). He has travel to give talks at ENS Lyons in September 2006, and in Pisa Italy in Nov 2006.

The PIs of the proposal have been invited to numerous international conferences to present their works. IPAM at UCLA is having a special program on the mass transportation during Spring 2008. W. Gangbo is one of the organizers of this program. L.C. Evans is organizing a workshop in the program. The other members of the FRG will be on site at IPAM for few weeks/months. They will interact with students and postdocs coming from the USA and all over the world.

The FRG supported a meeting in Edinburgh in Summer 2007. Based on the comments we got (comments forms were turned in) this was an extremely successful meeting.

Journal Publications

C. Evans, W. Gangbo, O. Savin., "Diffeomorphisms and Nonlinear Heat Flows.", SIAM Journal of Analysis, p. 737, vol. 37, no, (2005). Published,

M. Cullen, R. Douglas, M. Sewell and I. Roulstone, "Generalised semi-geostrophic theory on the sphere", Journal of Fluid Mechanics, p. 123, vol. 531, (2005). Published,

M. Cullen and M. Feldman, "Lagrangian solutions of semi-geostrophic equations in physical space", SIAM J Math Analysis, p. 1371, vol. 37, (2006). Published,

M. Cullen, M Wakefield, "Modelling the response of the atmosphere to equatorial forcing", International Journal of Numerical Methods in Fluids, p. 1345, vol. 47, (2005). Published,

W. Gangbo, V. Oliker, "Existence of Optimal Maps in the Reflector-type Problems", ESAIM : COCV, p. 93, vol. 13, no, (2007). Published,

B. Dacorogna, W. Gangbo, "Extension Theorems for Vector Valued Maps", Jour Math. Pures Appliquees 313--344, 2006, p. 313, vol. 85, 3, (2006). Published,

Jochen Denzler and Robert J. McCann, "Fast diffusion to self-similarity: complete spectrum, long time asymptotics, and numerology", Arch. Rational Mech. Anal., p. 301, vol. 175, (2005). Published,

Robert J. McCann and Adam Oberman. Appendix by Maxim Trokhimtchouk., "Exact semi-geostrophic flows in an elliptical ocean basin", Nonlinearity 1891--1922, p. 1891, vol. 17, (2004). Published,

Jose Carrillo, Robert J. McCann and Cedric Villani, "Contractions in the 2-Wasserstein length space and thermalization of granular media", Arch. Rational Mech. Anal., p. 217, vol. 179, (2006). Published,

Dario Cordero-Erausquin, Robert J. McCann and Michael Schmuckenschlaeger, "Prekopa-Leindler type inequalities on Riemannian manifolds, Jacobi fields, and optimal transport", Ann. Fac. Sci. Toulouse Math., p. 613, vol. 6, 15, (2006). Published,

Yong-Jung Kim and Robert J. McCann, "Potential theory and optimal convergence rates in fast nonlinear diffusion", J. Math. Pures Appl., p. 42, vol. 86, (2006). Published,

Yong-Jung Kim and Robert J. McCann, "Sharp decay rates for the fastest conservative diffusions", C.R. Acad. Sci. Paris Ser. I Math., p. 157, vol. 341, (2005). Published,

Luis Silvestre and Emmanuel Milakis, "Regularity for fully non linear equations with Neumann boundary data", Comm. Partial Differential Equations, p. 1227, vol. 31, (2006). Published,

L. Caffarelli and P. Souganidis, "A rate of convergence for monotone finite difference approximations to fully nonlinear, uniformly elliptic PDEs", Comm. Pure Appl. Math, p. , vol. 61, (2008). Published,

L. Ambrosio, W. Gangbo, "Hamiltonian ODE's in the Wasserstein space of probability measures.", Comm. Pures Appl. Math, p. 0018, vol. LXI, (2008). Published,

M. Cullen, W. Gangbo, G. Pisante, "Semigeostrophic equations discretized in reference and dual variables", Arch. Rat. Mech. Ana., p. 341, vol. 185 no, (2007). Published,

M. Chipot, W. Gangbo and B. Kawohl, "On some nonlocal variational problems", Analysis and Applications, p. 345, vol. 4, (2006). Published,

Robert J. MCCann and Dejan Slepcev, "Second-order asymptotics for the fast-diffusion equation", Int. Math. Res. Not., p. 1, vol. 24947, (2006). Published,

Luis A. Caffarelli and Robert J. McCann, "Free boundaries in optimal transport and Monge-Ampere obstacle problems", Annals of Math, p. ???, vol. 2, (2010). Accepted,

Gui-Qiang Chen, Mikhail Feldman, "Potential theory for shock reflection by a large-angle wedge", Proc. Nat. Acad. Sci. U.S.A., p. 15368, vol. 102, (2005). Published,

Gui-Qiang Chen, Mikhail Feldman, "Existence and Stability of Multidimensional Transonic Flows through an Infinite Nozzle of Arbitrary Cross-Sections", Arch. Ration. Mech. Anal., p. 185, vol. 184 no, (2007). Published,

Jochen Denzler, Robert McCann, "Nonlinear diffusion from a delocalized source: affine self-similarity, time reversal, & nonradial focusing geometries", Ann. Inst. H. Poincare Anal. Non Lineaire, p. 865, vol. 25, (2008). Published,

Robert McCann and Peter Topping, "Ricci flow, entropy and optimal transportation, with Peter Topping", Amer. J. Math, p. ??, vol. ????, (2010). Accepted,

L.C. Evans and O. Savin, "$C^{1, \alpha}$ regularity for infinity harmonic functions in two dimensions", Calc. Var. Partial Differential Equations, p. 325, vol. 32 no 3, (2008). Published,

W. Gangbo, T. Nguyen, A. Tudorascu, "Euler--Poisson systems as action-minimizing paths in the Wasserstein space", Archive Rational Mechanics Analysis, p. ??, vol. ???, (2008). Accepted,

G. Bouchitte, W. Gangbo and P. Seppecher, "Michell trusses and existence of lines of principal actions.", Mathematical Models and Methods in Applied Sciences, p. 1571, vol. 18, 9, (2008). Published,

Cullen,M.J.P., "Modelling atmospheric flows. Invited paper prepared for 2007 issue.", Acta Numerica, p. 67, vol. 16, (2007). Published,

Cullen,M.J.P., "Semigeostrophic solutions for flow over a ridge.", Quart. J. Roy. Meteorol. Soc., p. 491, vol. 133, is, (2007). Published,

Dorian Goldman and Robert McCann, "Chaotic response of the 2D semi-geostrophic and 3D quasi-geostrophic equations to gentle periodic forcing", Nonlinearity, p. 1455, vol. 21, (2008). Published,

R. McCann and Maxim Trokhimtchouk, "Optimal partition of a large labor force into working pairs", Econom. Theory, p. ??, vol. ???, (2010). Accepted,

Books or Other One-time Publications

Mike Cullen, "A mathematical theory of large-scale atmospheric flow", ( ). Book, Submitted
Editor(s): World Scientific Press
Bibliography: Nothing to report yet

L.C. Evans and M. Zworski, "Lecture notes on semiclassical analysis (quantize weak KAM theory)", ( ). Book, In progress
Editor(s): Not chosen yet
Bibliography: Nothing to report yet

Web/Internet Site

Other Specific Products

Contributions

Contributions within Discipline:

Bouchitte, Gangbo, Seppecher consider the new concept of Michell trusses they introduced as a first concrete step which support Michell theory of existence of Michell trusses. They anticipate that the probability appraoch in that paper could be useful in the study of hyperbolic equation.

With L. Ambrosio, W. Gangbo study existence of solutions in infinite dimensional Hamiltonian systems. With T. Pacini, W. Gangbo introduced a Poisson structure on the set of probability measures. With T. Nguyen and A. Tudorascu, W. Gangbo hopes to study invariant measures for PDEs that are infinite dimensional Hamiltonian systems.

McCann gave a series of 5 introductory lectures on Optimal Transportation and its Applications at the MSRI workshop November 14-18, 2005 sponsored by our FRG, and a series of 5 lectures on the Regularity theory of Optimal Maps to an audience of graduate students at the International Summer School on Partial Differential Equations at Hangzhou, China (May 23-June 3, 2005)

McCann gave more than a dozen lectures in the USA and Europe last year, including being a senior participabt in the Institute Henri Poincare program on Phenomena of Large Dimensions in May and July of 2006.

Last fall Evans ran a weekly student PDE seminar, during which my students gave 1 1/2 hour talks on a huge variety of topics in nonlinear

PDE. This semester he is doubling the seminar: they have one student 1 1/2 hour talk before lunch on Wednesdays and another after lunch.

Gangbo is running the PDE seminar at Georgia Tech for a second. The first year involved many students as a speaker. Most speakers during the second were top experts in their fields. Their expenses were supported by the FRG and other sources.

Contributions to Other Disciplines:

It remains yet unclear how the use of the weak notion of Michell trusses mentioned above, will be of great use to engineers in material sciences or elasticity theory. But, we are hopeful. We hope to better understand what invariant measures are for PDEs since we have a framework to approximate conservative PDEs by finite dimensional Hamiltonian systems. The infinite dimensional Hamiltonian approach would exploit what is known in the weak KAM theory to obtain results in PDEs.

Some of our studies are relevant in meteorology and oceanography. M. Cullen serves as a liaison between us and the Meteo Office, Exeter. He is paying attention to the consistency between their computations and the theory developed by him and various PIs of this proposal, such as Feldman, Gangbo or McCann.

Our works contribute to a better understanding of the kinetic theory of gases. We study problems on nonlinear diffusion, widely used as a model for oil recovery, population spreading, and thermal shock waves in plasmas; The asymptotics behaviour of the evolutive systems we study, serve as a model for equilibration rate problems in mathematical physics and kinetic theory.

In collaboration with P Souganidis, L. Caffarelli's work on on rate of convergence of solutions to fully non linear equations in random media to their ergodic limit, in case of a given mixing rate is a contribution to Ergodic theory abd Dynamical Systems.

L. C. Evans and many of his students have been working on the KAM theory, in connection with the Mass transport theory. This is a clear contribution to Dynamical Systems and Semiclassical Analysis.

McCann's work has contributed to at better understanding of SGS which appear in meteorology and oceanography. His work is useful in kinetic theory, including understanding nonlinear diffusion systems in a model for oil recovery, population spreading, and thermal shock waves in plasmas; The studies related to asymptotics serve as a model for equilibration rate problems in mathematical physics and kinetic theory.

Gangbo and his former postdoc A. Tudorascu made a progress in understanding how to extend the weak KAM theory to infinite dimensional spaces. They hope that approach will be of interest in dynamical system.

Contributions to Human Resource Development:

Nothing yet to report

Contributions to Resources for Research and Education:

Nothing yet to report

Contributions Beyond Science and Engineering:

Nothing yet to report

Conference Proceedings

Categories for which nothing is reported:

Organizational Partners

Any Web/Internet Site

Any Product

Any Conference

Progress Report: FRG Grant DMS-0354729
Lawrence C. Evans, Mathematics Department, UC Berkeley

A. Weak KAM theory.

The central issue here is understanding how the effective Hamiltonian $\bar{H}$, built from a given Hamiltonian H, encodes information about the corresponding Hamiltonian dynamics. In the current spring semester, 2005 Evans has been running a seminar in which my graduate students are lecturing in detail about KAM constructions: the objective is finding stronger connections with the weak KAM methods.

Evans's very good PhD student Y. Yu is now finishing his thesis, mostly on the *Aronsson equation*

$$A_H[u] := -\sum_{i,j=1}^{n} H_{p_i} H_{p_j} u_{x_i x_j} - \sum_{i=1}^{n} H_{p_i} H_{x_j} = 0,$$

where the nonlinearity H is evaluated at (Du, x), and its connections with weak KAM, especially the Mather and Aubry sets.

Mr. Yu has been supported this semester on funds remaining from our previous FRG grant. Next fall he will go for a postdoc at the University of Texas, to work with Caffarelli and Souganidis.

Evans is also still working with M. Zworski on a forthcoming set of lecture notes on semiclassical analysis. The eventual goal is understanding these methods well enough to "quantize weak KAM theory", if indeed this makes sense.

B. Lakes and rivers.

Inspired by ideas of L. Prigozhin, Evans has introduced the singular limit problem

$$\begin{cases} u^\epsilon_t - \operatorname{div}\left(\frac{\phi(u^\epsilon - g)}{\epsilon} D_x u^\epsilon\right) = f & \text{in } \mathbb{R}^2 \times (0,\infty). \\ u^\epsilon = g & \text{on } \mathbb{R}^2 \times \{t = 0\}. \end{cases}$$

Here g represents the height of a given landscape and $\phi(x) = 0$ if $x \le 0$, $\phi(x) > 0$ if $x > 0$. If $u^\epsilon \to u$ as $\epsilon \to 0$, then formally

$$|Du| = 0 \quad \text{on the set } \{u > g\}.$$

Think of f as a "rainfall" rate and u the height of the resulting "water flow" over the landscape. The components of the set $\{u > g\}$ are "lakes".

This is a form of the "sandpile" models introduced by Aronsson, Evans and Wu some years ago in connection with optimal mass transport. Evans has recently discovered how to use a variant of the Baiocchi

transform from free boundary theory to study the sets $\{u^\epsilon > g\}$, and hopes thereby to make rigorous some formal asymptotics J. Neu worked out.